A Systematic Approach to Deep Inference for Modal Logic

Sixth International Workshop on Structures and Deduction 2026

Justus Becker

University of Birmingham

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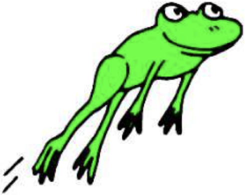
First, a preliminary warning

This is very much work in progress...



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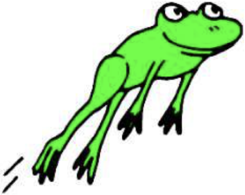

I will present more *ideas* and initial *observations*.

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Logical language (with $\mathrm{Pr} = \{a, b, c, ...\}$ a countable set of propositional atoms):

$$A ::= \bot \mid \top \mid a \mid \overline{a} \mid A \vee A \mid A \wedge A \mid \Box A \mid \Diamond A$$

Negation is defined via De Morgan duality and double negation elimination: $\overline{\overline{p}} = p$, $\overline{\bot} = \top$, $\overline{\top} = \bot$, $\overline{(A \vee B)} = \overline{A} \wedge \overline{B}$, $\overline{(A \wedge B)} = \overline{A} \vee \overline{B}$, $\overline{\Box A} = \Diamond \overline{A}$, and $\overline{\Diamond A} = \Box \overline{A}$.
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Axiomatisation of modal logic K:

$$\mathsf{K} := \mathsf{CPL} + \Box(A \to B) \to (\Box A \to \Box B) + \frac{A}{\Box A}$$

Naturally, CPL includes the rule of modus ponens $\dfrac{A \to B \quad A}{B}$.

Proof Theory

Deep Inference

On a high level *Deep Inference* builds on the idea that deductions should have as little additional syntax as possible.
This demands deduction *deep* inside formulas. For example, we might have a derivation like this:

$$\dfrac{\dfrac{A \wedge \top \wedge C}{A \wedge (B \vee \overline{B}) \wedge C}}{A \wedge ((B \wedge \overline{B}) \vee C)}$$

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This formalism of "derivations as lists" allows us to manipulate derivations in a unique way. E.g. decomposing proofs into phases or contrapositively inverting derivations.

$$\begin{array}{c} \overline{B} \\ \Big\Vert\, S{\uparrow} \\ \overline{A} \end{array} \quad \leftrightsquigarrow \quad \begin{array}{c} A \\ \Big\Vert\, S{\downarrow} \\ B \end{array} \quad \rightsquigarrow \quad \begin{array}{c} A \\ \Vert\, S_2 \\ C \\ \Vert\, S_1 \\ B \end{array}$$

Open Deduction

While in the *Calculus of Structure* derivations are lists of formulas, in *Open Deduction* we can apply rules in parallel.
Consider for example, the two derivations which only differ by the order of rule applications.

$$\cfrac{\cfrac{A \wedge B}{C \wedge B}}{C \wedge D} \qquad \cfrac{\cfrac{A \wedge B}{A \wedge D}}{C \wedge D}$$

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Thus we define derivations by

$$\mathcal{D} ::= \bot \mid \top \mid p \mid \overline{p} \mid \mathcal{D} \vee \mathcal{D} \mid \mathcal{D} \wedge \mathcal{D} \mid \Box\mathcal{D} \mid \Diamond\mathcal{D} \mid \rho\,\dfrac{\mathcal{D}}{\mathcal{D}}$$

where ρ is a rule. Proofs are derivations starting with $\top$.

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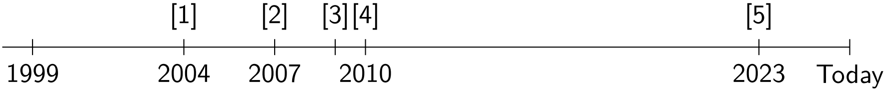
- We can separate operational and modal reasoning. We might ask "How much does theorem A rely on modal vs. non-modal reasoning?"
- Many previous results do not agree on what the Deep Inference system for K is, and factor through other proof formalisms

History of Deep Inference for Modal Logic



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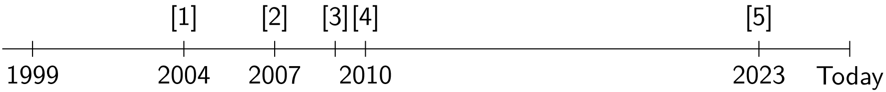

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Symmetric (K)lassical System

We build upon the well-known calculus SKS for classical propositional logic and its cut-free version KS.

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Structural Rules:

$$\mathsf{i}{\downarrow}\,\frac{\top}{A \vee \bar{A}} \qquad \mathsf{i}{\uparrow}\,\frac{A \wedge \bar{A}}{\bot} \qquad \mathsf{w}{\downarrow}\,\frac{\bot}{A} \qquad \mathsf{w}{\uparrow}\,\frac{A}{\top} \qquad \mathsf{c}{\downarrow}\,\frac{A \vee A}{A} \qquad \mathsf{c}{\uparrow}\,\frac{A}{A \wedge A}$$

Switch and Medial:

$$\mathsf{s}\,\frac{(A \vee C) \wedge B}{(A \wedge B) \vee C} \qquad \mathsf{m}\,\frac{(A \wedge C) \vee (B \wedge D)}{(A \vee B) \wedge (C \vee D)}$$

Equational Theory:

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C \qquad A \wedge B = B \wedge A \qquad A \wedge \top = A$$

$$A \vee (B \vee C) = (A \vee B) \vee C \qquad A \vee B = B \vee A \qquad A \vee \bot = A$$

A general approach to designing rules

"Everything" is a medial!

$$\downarrow \frac{(A \star C) \bullet (B \star D)}{(A \bullet B) \star (C \circ D)}$$

NB: The connectives $\bullet, \circ, \star$ might be of any arity.

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What about $\star = \top$ and $\bullet = \Diamond$? This yields the rule $\dfrac{\Diamond \top}{\top}$ which is also sound in K.

SKS(K)

Doing this "analysing all medials" and also dualising all rules gives us the following additional rules.

$$\mathsf{k}\!\downarrow \frac{\Box(A \vee B)}{\Box A \vee \Diamond B} \qquad \mathsf{k}\!\uparrow \frac{\Diamond A \wedge \Box B}{\Diamond(A \wedge B)}$$

$$\Box\mathsf{w}\!\downarrow \frac{\bot}{\Box\bot} \qquad \Diamond\mathsf{w}\!\uparrow \frac{\Diamond\top}{\top}$$

$$\Box\!\downarrow \frac{\Box A \vee \Box B}{\Box(A \vee B)} \qquad \Diamond\!\uparrow \frac{\Diamond(A \wedge B)}{\Diamond A \wedge \Diamond B}$$

$$\Box\top = \top \qquad \Diamond\bot = \bot$$

$$\Box(A \wedge B) = \Box A \wedge \Box B \qquad \Diamond(A \vee B) = \Diamond A \vee \Diamond B$$

We define SKS together with these rules as SKS(K).
Note the duality between the left and right sides!

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By the soundness of our rules and the cut rule we immediately have soundness and completeness.

Theorem

$A \vdash_{\mathsf{K}} B$ *iff* $A \vdash_{\mathsf{SKS(K)}} B$

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Theorem

$A \vdash_{\mathsf{K}} B$ *iff* $A \vdash_{\mathsf{SKS(K)}} B$

We can separate out multiple systems of SKS(K) which are still sound and complete.
This includes a cut-free system KS(K) which is complete by virtue of previous results (Ideally, we want a proof of this using only techniques of Deep Inference).

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A strict non-atomic system

Conversely, we can use (non-atomic) structural rules to derive other rules. Consider the equation $\Box(A \wedge B) = \Box A \wedge \Box B$. Using (non-atomic) contraction and weakening, we can derive one of the directions:

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Similarly, we can remove $\Box\mathsf{w}\downarrow, \Diamond\mathsf{w}\uparrow, \Box\downarrow, \Diamond\uparrow$ as well as reduce further equations to rules.

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This leaves us the modal rules $\mathsf{k}\downarrow\,\dfrac{\Box(A \vee B)}{\Box A \vee \Diamond B}$, $\mathit{nec}\downarrow\,\dfrac{\top}{\Box\top}$, $\Box\uparrow\,\dfrac{\Box A \wedge \Box B}{\Box(A \wedge B)}$ and its duals.

Going further

In this talk I...

- defined a proof system for modal logic K in the open deduction formalism
- obtained soundness and completeness for this system
- showed that it admits cut-elimination (factoring through previous results)
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Thank You!